NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS OF SECOND ORDER

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Abstract– we consider the nonlinear elliptic boundary value problems of second order. We proved that monotone solution of nonlinear elliptic boundary value problems of second order can be obtained. We also investigated the stability of solution obtained by the monotone method and found it to be stable, also that solutions obtained (that is upper and lower solutions) are maximal and minimal solutions.

Index Terms— Geometry,nonlinear,elliptic,boundary

1 INTRODUCTION

Partial differential equation is well established area of mathematics and it plays an important role in modern mathematics especially in analysis and geometry [2].

The theory of nonlinear elliptic equations of arbitrary order is nowadays of the actively developing branches of the theory of partial differential equations [3]. These types of elliptic problems arise in various fields of applied sciences [1].

Gilbarg D., Trudinger, N. S., Agmon S., Nussbaum R. worked on nonlinear elliptic boundary value problems of second order. The monotone method and its associated upper and lower solutions for nonlinear partial differential equations have been given extensive attention in recent years. The method is popular because not only does it give constructive proof for existence theorem but it also leads to various comparism results which are effective tools for the study of qualitative properties of solutions. The monotone behaviour of the sequence of iterations is also useful in the treatment of numerical solutions of various boundary value problems.

In this paper, we developed the monotone method in the presence of lower and upper solutions for the problem

$$Lu + f(x,u) = 0 \ in \ \Omega \qquad (1)$$

$$Bu = g \ on \ \partial\Omega$$

Where L is a second order uniformly elliptic operator:

$$L = \sum a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i(x)\frac{\partial}{\partial x_i}$$

a_{ij}, b_i are functions which are assumed to be smooth, and B is one of the boundary operators

$$Bu = u$$

or

$$Bu = \frac{\partial u}{\partial \nu} + \beta(x)u, \qquad x \in \partial\Omega$$

Here $\frac{\partial}{\partial \nu}$ denotes the outward conormal derivative, and we assume $\beta \geq 0$ everywhere on the boundary, $\partial\Omega$ of Ω. The coefficients of the operator L are assumed to be Hölder's continuous and the matrix (a_{ij}) is uniformly positive definite on $\bar{\Omega}$. Recognizing the immense value to nonlinear problems, we apply this monotone method using the idea of upper and lower solutions to second as the order nonlinear partial differential equations.

2 METHODS

In this section, we treat the techniques for the solvability of uniformly elliptic second order boundary value problems which are the monotone iteration schemes and maximum principles.

2.1 MONOTONE ITERATIVE SCHEME

Monotone iterative scheme is a mathematical algorithm used to construct a solution of nonlinear boundary value problem. This scheme is used in such a way that an initial approximation is used to calculate the second approximation, which in turn is used to calculate the third and so on. The limit of the sequence constructed converges to a solution of the differential equation.

The monotone iterative scheme is also known as the method of upper and lower solutions. The basic idea of this method is that by using the upper or lower solution as the initial iteration in a suitable iterative process, the resulting sequence of iteration is monotone and will converge to a solution of the problem.

Now, let us consider the following nonlinear elliptic boundary value problem

$Lu = f(x,u) \; in \; \Omega$

$Bu = g(x) \; on \; \partial\Omega$

Where Ω is a boundary domain in $\mathbb{R}^n \; (n = 1,2,\ldots), \partial\Omega$ is the boundary of Ω, B is a boundary operator and L is elliptic operator which is defined as

$L = \sum a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i(x)\frac{\partial}{\partial x_i}, x = (x_1, x_2, \ldots, x_n)$ (5)

To determine the existence of the solution of the boundary value problem (4), we shall consider the following hypothesis and theorem:

Hypothesis 2.1 (i) The function $K(x)$ is continuous nonnegative in $\bar{\Omega}$ and possesses the following property:

$K_0 = \int K(x)dx \leq 1$ (6)

(ii) $f(x,0) \geq 0$ and there exists a constant $m_0 > 0$ such that $f(x,u)$ is a C^1-function in u and $f_v(x,u) \geq 0$ for $u, v \in [0, m_0]$

Theorem 2.1 Let ϕ be a supersolution and ψ a subsolution of (4) with $\psi \leq \phi$. Then there exists solution u and $\bar{u}$ with

$\psi \leq u \leq \bar{u} \leq \phi$

where u is the minimum solution between ψ and ϕ, and $\bar{u}$ is the maximum solution between ψ and ϕ, u may be equal to $\bar{u}$,

In theorem 2.1, supersolution and subsolution stand for upper and lower solutions respectively.

2.2 Construction of Monotone Sequence

To construct a monotone sequence, we suppose f satisfies one-sided Lipschitz condition

$f(x,u_2) - f(x,u_1) \geq -K(x)(u_2 - u_1)$ for $\phi \leq u_1 \leq u_2 \leq \psi$ **(7)**

where k is a bounded nonnegative function in Ω.

If we add ku on both sides of (7), we have

$F(x,u) = k(x) + f(x,u)$

Now if we let $Lu = -\Delta u$, then we have

$-\Delta u + ku = F(x,u)$ in Ω (4)

(9)

$Bu = g(x)$ on $\partial\Omega$

By condition (7), $F(x,u)$ is monotone nondecreasing in u for $u \in [\phi, \psi]$. We assume $c \in C^\alpha(\bar{\Omega})$, so that $F(x,u)$ is Hölder continuous in $\bar{\Omega} \times [\phi, \psi]$. Hence for any given $u^0 \in C^\alpha(\bar{\Omega})$, one can construct a sequence $\{u^{(k)}\}$ from the following iteration process, for $k = 1, 2, \ldots$

$Lu^{(k)} + ku^{(k)} = F(x, u^{(k-1)})$ in Ω

(10)

$Bu^{(k)} = g(x)$

The sequence $\{\bar{u}^{(k)}\}$ is called the upper sequence if the iteration $u^{(0)} = \psi$ and the sequence $\{u^{(k)}\}$ is called the lower sequence if the initial iteration $u^{(0)} = \phi$

2.3 Maximum Principles

The most important tool that investigates most properties of solutions and equations of second order elliptic and parabolic type is the maximum principles. This has types and all these types enable us to obtain valuable information about the properties of solutions and perhaps the equations themselves.

Theorem 2.2: Weak Maximum principles

Consider a uniform elliptic operator of the form (2). Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and let $u(x,y) \in C^2(\Omega) \cap C(\Omega)$ be a harmonic function in Ω (be the solution of the equation $Lu = f$, where $f \in C(\Omega)$ and $f \leq 0$ in Ω). Then, the maximum of $u \in \bar{\Omega}$ is obtained on the boundary $\partial\Omega$.

Theorem 2.3: Strong Maximum Principle
Let Ω be a bounded open set that is connected. Let u be a continuous function on $\bar{\Omega}$ that is a solution of the elliptic equation in Ω. Let M be the maximum value of u, if there is a point $x \in \Omega$ such that $u(x) = M$, then u is constant in Ω.

Corollary 2.4
Let L be a uniformly elliptic operator of the form:
$Lu = a_{ij}(x)u_{ij} + b_k(x) + c(x)u$
with $c \leq 0$, and $u, v \in C^2(\Omega) \cap C^0(\Omega)$ satisfying
$Lu \geq Lv$,
$u|_{\partial\Omega} \leq v|_{\partial\Omega}$

Then for all interior points $\in \Omega$, we have
$u(x) < v(x)$.

3 RESULTS AND DISCUSSION

This section shows how solutions of non-linear elliptic second order boundary value problems are constructed using monotone methods.

3.1 Construction of Solutions by Monotone Methods

In this section, we discuss the use of monotone iteration schemes in the construction of non-linear elliptic boundary value problems of the type (1) – (3). This method is in the presence of lower and upper solutions for this problem (1).

Theorem 3.1 Let there exist two smooth functions U_0 and V_0, where $U_0 \geq V_0$, such that

$$Lu_0 + f(x, u_0) \leq 0,$$
$$Bu_0 \geq g$$

and
$Lv_0 + f(x, v_0) \geq 0$
$Bv_0 \leq g$
Assume f is a smooth function on
$\min v_0 \leq u \leq \max u_0$
Then there exist a regular (smooth) solution of
$Lw + f(x, w) = 0$ (13)
$Bw = g$
such that

$v_0 \leq w \leq u_0$

Proof

We assume that $\frac{\partial f}{\partial u}(x, u)$ is bounded below for $x \in \Omega$, and $\min v_0 \leq u \leq \max u_0$, so that

$\frac{\partial f}{\partial u}(x, u) + \eta u > 0$

for all $x \in \Omega$, and u in that interval, provide $\boldsymbol{\eta}$ is a sufficiently large operator. We define the mapping T as

$Tu = v$ if (11)

$$(L - \eta)v = -[f(x, u) + \eta u], \quad (12) \quad (15)$$
$$Bv = g.$$

T is a completely continuous mapping (operator) since it takes C^α into $C^{2+\alpha}$ by the Schauder estimates (regularity theorem) for elliptic equations.
The operator T is also monotone in the sense that,
$u \leq v$ implies $Tu < Tv$
Provided u and v are restricted to the set
$\min v_0 \leq u,\ v \leq \max u_0$.
Then,

$$(L - \eta)Tu = -[f(x, u) + \eta u],$$
$$BTu = g$$

and
$(L - \eta)Tv = -[f(x, u) + \eta v$
$BTV = g$.
Therefore
$(L - \eta)(Tv - Tu) = -[f(x, v) - f(x, u) + \eta(v - u)],$ (16)
$B(Tv - Tu) = 0$
Hence, since $u \leq v$, the quantity is non-negative.
Let us define $F(x, u) = f(x, u) + \eta u$.
Then $Fu > 0$ implies that F is increasing. Therefore the right hand side is
$F(x, v) - F(x, u)$
So
$(L - \eta)w \leq 0,$
$Bw = 0$
where
$w = Tv - Tu.$
By the strong maximum principle (Theorem 2.3) for elliptic operators,
$w > 0$ in Ω (unless $w = 0$ in which case $Tu = Tv$ and the right hand side of (16) is identically zero), but since $u \neq v$ and F is strictly monotone, we define $u_1 = Tu_0$ and $v_1 = Tv_0$.

Let us then show that
$u_1 < u_0$ and $v_1 > v_0$
we have
$(L - \eta)u_1 = -[f(x,u_0) + \eta u_0],$
$Bu_1 = g,$
So
$(L - \eta)(u_1 - u_0) = -[f(x,u_0) - \eta u_0 - Lu_0 + \eta u_0$ (18)
$= m - [Lu_0 + f(x,u_0)] \geq 0$

and
$B(u_1 - u_0) = g - Bu_0 \leq 0$
Again by theorem 2.3, we have
$u_1 < u_0$ where $Lu_0 + f(x,u) \neq 0$
Similarly $v_1 > v_0$ since
$u_1 < u_0, Tu_1 < Tu_0 = u_1$. Thus, the sequence defined inductively by $u_1 = Tu_0, u_2 = Tu_1, u_n = Tu_{n-1}$ is monotone decreasing.
Similarly, $v_n = Tv_{n-1}, v_2 = Tv_1, v_1 = Tv_0$ defines a monotone increasing sequence. Furthermore, we have
$v_n < u_n \; \forall n:$
$v_0 < v_1 < v_2 < \cdots < v_n < \cdots u_n < u_{n-1} < \cdots u_1 < u_0$ (19)

Infact,
$v_0 < u_0,$
Suppose $v_{n-1} < u_{n-1}$
then $u_n = Tu_{n-1} > Tv_{n-1} = v_n,$
so the proof follows by induction since the sequences $\{u_k\}$ and $\{v_k\}$ are monotone, the pointwise limits
$\bar{u}(x) = \lim u_k(x) \; and \; \bar{v}(x) = \lim v_k(x)$
exists. The operator T is a composition of the nonlinear operation
$u \to f(x,u) + \eta u$
with the inversion of the linear, inhomogeneous elliptic boundary value problem $\varphi \to v$ defined by
$(L - \eta)v = \varphi \; on \; \Omega$ (22)
$Bv = g \; on \; \partial\Omega$
For u bounded and $f(x,u)$ bounded on the range of u, the first operation takes bounded pointwise convergent sequences into pointwise convergent sequences. The operation $\varphi \to v$ takes $L_n(\Omega)$ continuously into the Sobolev space $W^{2,p}(\Omega)$ for all $p, 1 < p < \infty$ by the L_n estimates of Agmon, Douglis and Nirenberg.

Thus, since $u_k = Tu_k$ and since $\{u_k\}$ is a bounded, pointwise convergent sequence, it converges also in the Sobolev space $W^{2,p}$. Hence $W^{2,p}(\Omega)$ is continuously embedded into $C^{1+\alpha}$ for $\alpha = 1 - \frac{n}{p}$, when $p > n$. Therefore $\{u_k\}$ converges in $C^{1+\alpha}$, and by the Classical Schauder estimates for regular elliptic boundary value problems, $\{u_k\}$ then also converges in $C^{2+\alpha}$, and by the Classical estimates for regular elliptic boundary value problems, $\{u_k\}$ then also converges in $C^{2+\alpha}$.
We thus have,
$\bar{u} = \lim u_k = \lim Tu_{k-1} = T \lim u_{k-1} = T\bar{u}$

and similarly for $\bar{v}$, by the continuity of T.
Thus $\bar{u}$ and $\bar{v}$ are fixed points of T, and furthermore, they are of class $C^{2+\alpha}(\Omega)$ for $0 < \alpha < 1$.
They are therefore regular solutions of the elliptic boundary value problem (1).

Corollary 3.2 The solutions $\bar{u}$ and $\bar{v}$ of problem (1) are maximal and minimal solutions in the region
$v_0 \leq u \leq u_0$
That is, if w is any solution of (1) such that
$v_0 \leq w \leq u_0,$ (20)
then
$\bar{u} \leq w \leq \bar{v}.$

Proof
We have $w = Tw, u_1 = Tu_0;$
since
$w \leq u_0 \; Tw < Tu_0,$ or
$w < u_1$ (21)
By induction,
$w \leq u_n \; \forall n; n = 0, 1, 2, \ldots$
hence
$w \leq \bar{u}(x).$
Similarly,

$w \geq \bar{v}(x).$

Conclusion

The paper is affirmative and we accomplished the aim of using the monotone methods in constructing solutions of nonlinear elliptic boundary value problems of second order. The anomalies observed in previous work which required that monotonicity, positivity, convexity or boundedness are conditions of the func-

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tion $f(x,u)$ are corrected in this paper which only required $f(x,u)$ to be smooth.

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